



Introduction to Probability and Statistics

Slides 2 – Chapter 2

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Chapter 2

Probability

Chapter Goals

After completing this chapter, you should be able to:

- Explain three approaches to assessing probabilities
- Apply common rules of probability, including the Addition Rule and the Multiplication Rule
- Use Bayes' Theorem for conditional probabilities

Important Terms

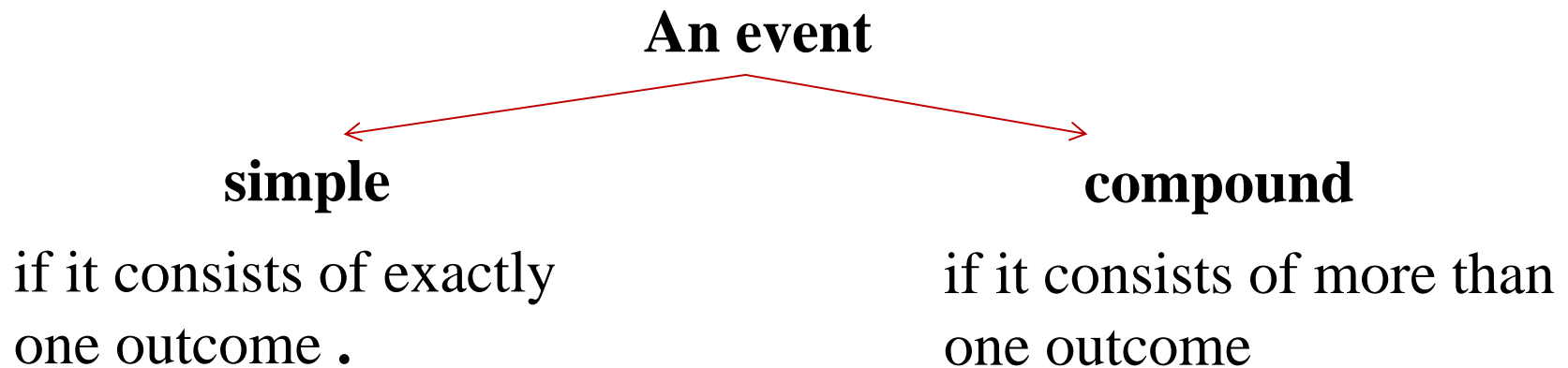
- Probability – the chance that an uncertain event will occur (**always between 0 and 1**)
- Experiment – a process that produces outcomes for uncertain events
- Sample Space – the collection of all possible experimental outcomes
- Event – a subset of sample space.

2.1 Sample Spaces and Events

An Experiment: is any action or process whose outcome is subject to uncertainty.

The **sample space** of an experiment, denoted S (or Ω), is the set of all possible outcomes of that experiment.

An **event** A is a subset of the sample space S . That is $A \subseteq S$.



We say that an event A **occurs** if the outcome (the result) of the experiment is an element of A .

- $\varnothing \subseteq S$ is an event (\varnothing is called the impossible (**null**) event)
- $S \subseteq S$ is an event (S is called the sure event)

Example

Example: Rolling a die



The experiment has 6 possible outcomes.

Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

The following are possible events:

- $A = \{1, 3, 5\}$ getting an odd number.
 - $B = \{2, 4, 6\}$ getting an even number.
 - $C = \{1, 2, 3, 4\}$ getting a number less than 5.
 - $E = \{1, 3\}$ getting 1 or 3.
 - $S = \{1, 2, 3, 4, 5, 6\}$ getting a number < 10 .
 - $D = \{1\}$ getting number 1.
 - $F = \{8\} = \varnothing$ getting number 8.
- compound events
- sure event
- simple event
- Impossible event

Relations from Set Theory

An event is just a set, so relationships and results from elementary set theory can be used to study events.

- 1 The **union** of two events A and B is the event consisting of all outcomes that are either in A or in B .

Notation: $A \cup B$

Read: A or B .

$$A \cup B = \{x \in S: x \in A \text{ or } x \in B\}$$

 $A \cup B$ occurs if at least one of A and B occurs.

- 2 The **intersection** of two events A and B is the event consisting of all outcomes that are in both A and B .

Notation: $A \cap B$

Read: A and B .

$$A \cap B = AB = \{x \in S: x \in A \text{ and } x \in B\}$$

 $A \cap B$ occurs if both A and B occur together.

- 3 The **complement** of an event A is the set of all outcomes in S that are not contained in A .

Notation: A'

Read: not A .

$A' = \{x \in S: x \notin A\}$  A' occurs if A does not.

Example: Rolling a die.

The sample space: $S = \{1, 2, 3, 4, 5, 6\}$

Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{4, 6\}$. Then

$$A \cap B = \{1, 3\} \quad (A \cap B)' = \{2, 4, 5, 6\}$$

$$A \cup B = \{1, 2, 3, 5\} \quad (A \cup B)' = \{4, 6\}$$

$$A' = \{4, 5, 6\} \quad A' \cup B' = \{2, 4, 5, 6\}$$

$$B' = \{2, 4, 6\} \quad A' \cap B' = \{4, 6\}$$

$$A' \cap A = \varnothing$$

are equal

are equal

Mutually Exclusive

When A and B have no outcomes in common, $A \cap B = \varnothing$, they are **mutually exclusive** or **disjoint events**.

$A \cap C = \varnothing \longrightarrow A$ and C are **mutually exclusive (disjoint) events**.

Notice: The two disjoint event *do not occur together*.

Example:

Experiment: Selecting 3 items from manufacturing process; each item is inspected and classified as defective (**D**) or non-defective (**N**).

This experiment has $2^3 = 8$ possible outcomes. The **sample space** is
 $S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$

Consider the following events:

$A = \{\text{at least 2 defectives}\}$
 $= \{DDD, DDN, DND, NDD\} \subseteq S$

$B = \{\text{at most one defective}\}$
 $= \{DNN, NDN, NND, NNN\} \subseteq S$

$C = \{3 \text{ defectives}\} = \{DDD\} \subseteq S$

Then:

$A \cap B = \varnothing \rightarrow A \text{ and } B \text{ are } \textit{disjoint}.$

$A \cap C = \{DDD\} = C \rightarrow A \text{ \& } C \text{ are not } \textit{disjoint}.$

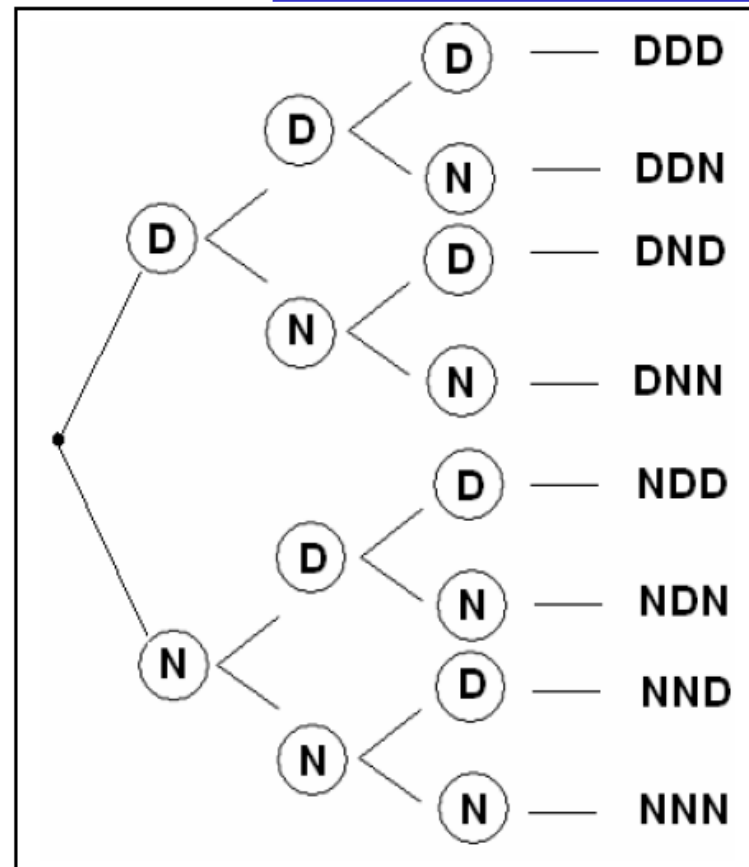
$B \cap C = \varnothing \rightarrow B \text{ and } C \text{ are } \textit{disjoint}.$

$A' = B$

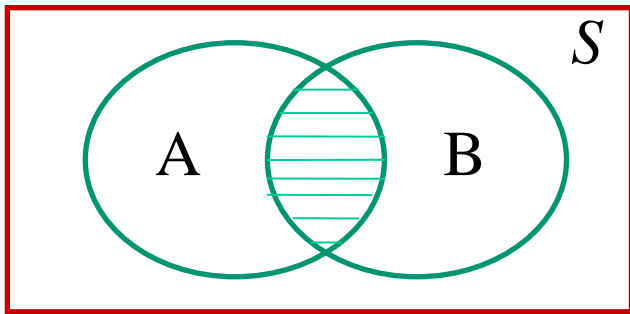
$B' = A$

$A \cup B = S$

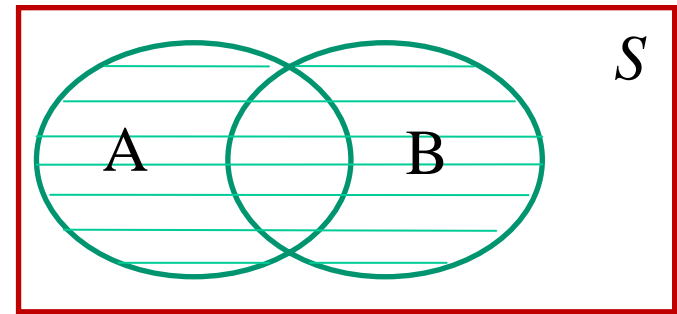
sample space S



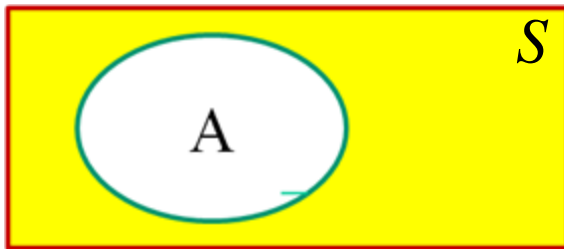
Venn diagram



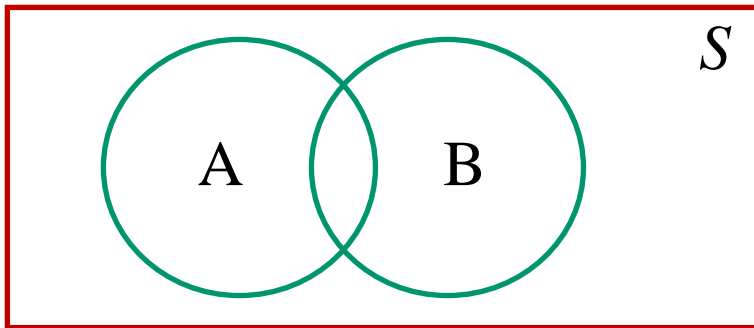
$$A \cap B$$



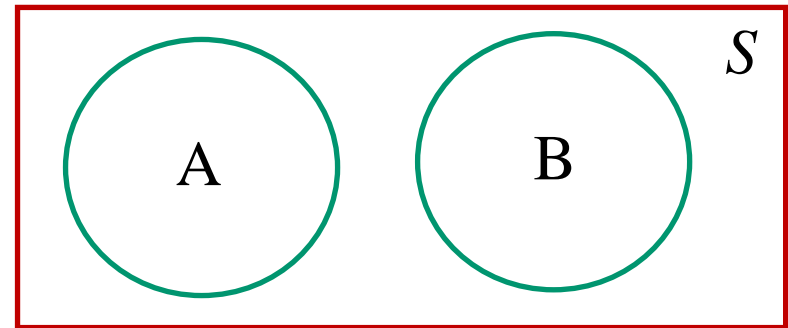
$$A \cup B$$



$$A'$$



A and B are not *mutually exclusive*.



A and B are *mutually exclusive*.

2.2 Axioms, Interpretations, and Properties of Probability

Given an experiment and a sample space S , the *objective of probability* is assign to each event A a number $P(A)$, called the probability of the event A , which will give *a precise measure of the chance that A will occur*.

Axioms of Probability

Axiom 1 $P(A) \geq 0$ for any event A .

Axiom 2 $P(S) = 1$.

Axiom 3 If A_1, A_2, \dots, A_k are disjoint events, then

$$P(A_1 \cup A_2 \cup \dots \cup A_k) = \sum_{i=1}^k P(A_i) \quad \text{finite set}$$

$$P(A_1 \cup A_2 \cup \dots) = \sum_{i=1}^{\infty} P(A_i) \quad \text{infinite set}$$

Properties of Probability

1 For any event A , $P(A) = 1 - P(A')$.

Proof: Since $S = A \cup A'$, then $1 = P(S) = P(A \cup A')$
 $1 = P(S) = P(A) + P(A') \longrightarrow P(A) = 1 - P(A')$.

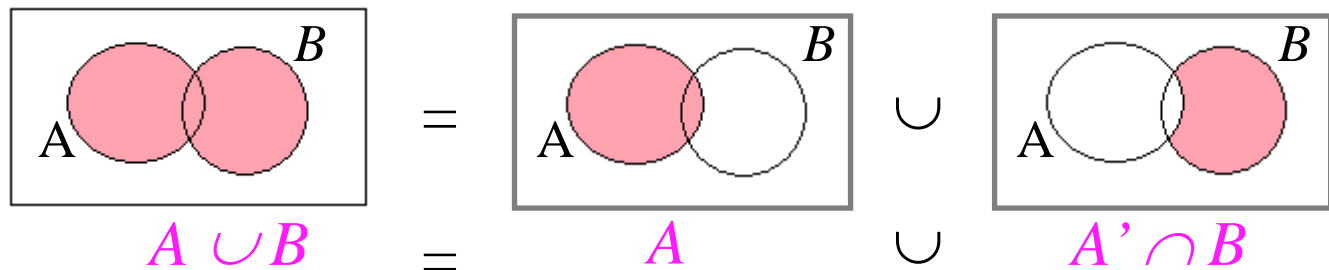
2 For any event A , $P(A) \leq 1$.

Proof: Since $1 = P(A) + P(A') \geq P(A)$ since $P(A') \geq 0$.

3 If A and B are mutually exclusive, then $P(A \cap B) = 0$.

4 For any two events A and B , $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof:



$P(A \cup B) = P(A) + P(A' \cap B)$, but $B = (A \cap B) \cup (A' \cap B)$ then
 $P(B) = P(A \cap B) + P(A' \cap B) \longrightarrow P(A' \cap B) = P(B) - P(A \cap B) \longrightarrow$
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example: In a certain residential suburb, 60% of all households subscribe to the metropolitan newspaper published in a nearby city, 80% subscribe to the local paper, and 50% of the households subscribe to both papers.

If a household is selected at random, what is the probability that it subscribes to (1) at least one of the two newspapers, (2) exactly one of the two newspapers ?

Solution:

Let $A = \{\text{subscribes to the metropolitan newspaper}\}$ and

$B = \{\text{subscribes to the local paper}\}$. Then

$$P(A) = 0.6, \quad P(B) = 0.8 \text{ and } P(A \cap B) = 0.5$$

$$(1) P(\text{at least one of the two newspapers}) = P(A \cup B)$$

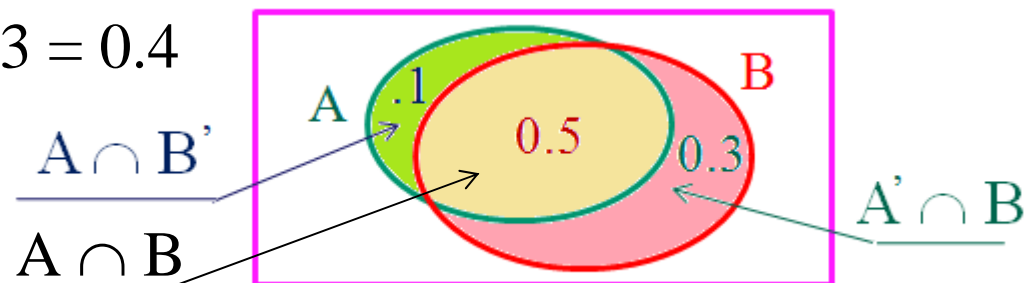
$$= P(A) + P(B) - P(A \cap B) = 0.6 + 0.8 - 0.5 = 0.9$$

$$(2) P(\text{exactly one}) = P(A \cap B') + P(A' \cap B)$$

$$P(A) = P(A \cap B') + P(A \cap B) \Rightarrow P(A \cap B') = 0.6 - 0.5 = 0.1$$

$$P(B) = P(A' \cap B) + P(A \cap B) \Rightarrow P(A' \cap B) = 0.8 - 0.5 = 0.3$$

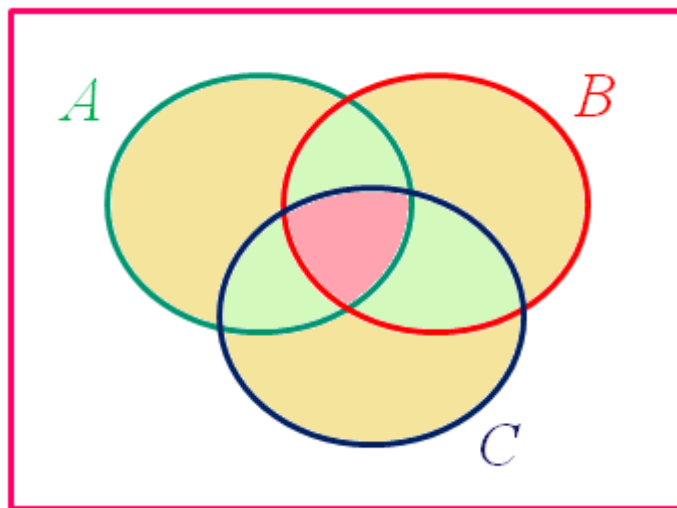
$$\Rightarrow P(\text{exactly one}) = 0.1 + 0.3 = 0.4$$



Probability of union of more than two events

For any three events A , B and C ,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$



Important laws will be given in the class

Problem 2.13

A computer consulting firm presently has bids out on three projects. Let $A_i = \{\text{awarded project } i\}$, $i=1,2,3$, and suppose that $P(A_1) = 0.22$, $P(A_2) = 0.25$, $P(A_3) = 0.28$, $P(A_1A_2) = 0.11$, $P(A_1A_3) = 0.05$, $P(A_2A_3) = 0.07$, and $P(A_1A_2A_3) = 0.01$.

Express in words each of the following events, and compute the probability of each event:

- a) $A_1 \cup A_2$ “Awarded either 1 or 2 or both” OR “awarded at least one”
- b) $A_1^c A_2^c$ “Awarded neither 1 nor 2”
- c) $A_1 \cup A_2 \cup A_3$ “Awarded either 1 or 2 or 3” OR “at least one of them”
- d) $A_1^c A_2^c A_3^c$ “Awarded none of the three projects”
- e) $A_1^c A_2^c A_3$ “Awarded # 3 but not # 1 nor # 2”
- f) $(A_1^c A_2^c) \cup A_3$ “either (neither # 1 nor # 2) or # 3”

Equally Likely Outcomes

If an experiment has $N(S)=N$ equally likely different outcomes, then **the probability of the event A** is:

$$P(A) = \frac{N(A)}{N}$$

where $N(A)$ denotes the number of outcomes contained in A .

Example: Rolling two fair dice separately.

What is the probability that sum of the two numbers is 7?

Solution:

$$N(S)=36$$

Let $A = \{\text{sum of two numbers} = 7\}$.

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$N(A) = 6$$

$$P(A) = \frac{6}{36} = \frac{1}{6}$$

The **sample space** is S

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

2.3 Counting Techniques

There are many counting techniques which can be used to count the number of points in the sample space (or in some events) without listing each element.

Product Rule

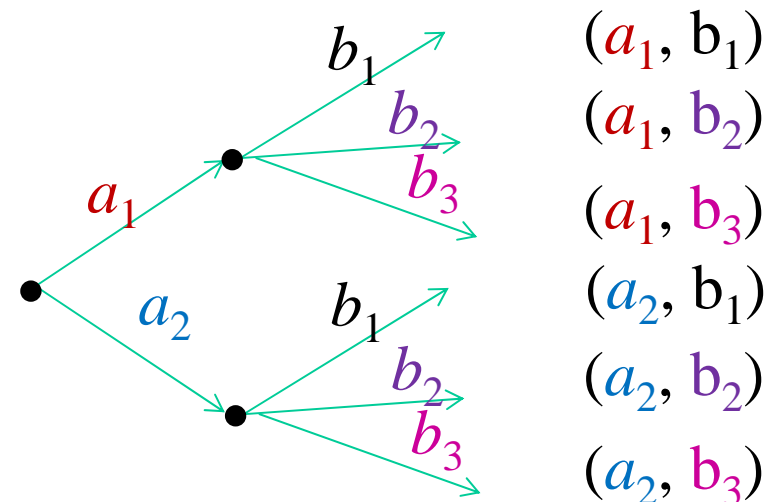
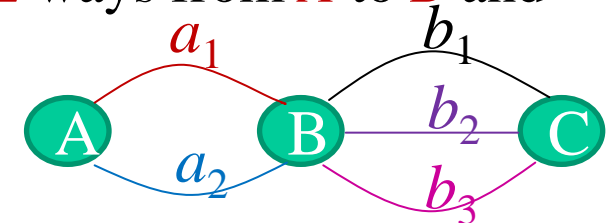
If the first element or object of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second can be selected in n_2 ways, then **the number of pairs** is $n_1 n_2$.

Example: Moving from City A to City C can be made in two steps. First moving from A to B then from B to C. There are $n_1=2$ ways from A to B and $n_2=3$ ways from B to C.

Then there are $n_1 \times n_2 = 2 \times 3 = 6$ ways from A to C.

These methods can be represented by the following pairs:

$(a_1, b_1), (a_1, b_2), (a_1, b_3),$
 $(a_2, b_1), (a_2, b_2), (a_2, b_3)$



Permutations

Any ordered sequence of k objects taken from a set of n **distinct objects** is called a **permutation of size k from n objects**.

Notation: $P_{k,n}$

$$P_{k,n} = n(n-1)(n-2) \dots (n-k+1)$$

Factorial

For any positive integer m , $m!$ is read “ m factorial” and is defined by $m! = m \times (m-1) \times (m-2) \dots \times (2) (1)$

Also, $0! = 1$.



$$P_{k,n} = \frac{n!}{(n-k)!}$$

Example

Example (2.21): There are 10 teaching assistants available for grading papers in a Statistics course at a large university. The first exam consists of 4 questions, and the professor wishes to select a different assistant to grade each question (*only one assistant per question*).

In how many ways can the assistants be chosen for grading?

Answer:

Here n = group size = 10, and
 k = subset size = 4.

The number of permutation is

$$P_{4, 10} = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10(9)(8)(7) = 5040$$

That is, there are 5040 different ways.

Example

Example: A boy has 4 beads – red, white, blue, and yellow. **How different ways can three of the beads be strung together in a row?**



This is a permutation of size 3 from 4 objects, since the beads will be in a row (**ordered**). Then the number is

$$P_{3,4} = \frac{4!}{(4-3)!} = 4! = 24$$

A diagram with two red boxes on the left. The top box contains the text "number selected" and the bottom box contains the text "total". Red lines connect the "3" in the subscript of $P_{3,4}$ to the "number selected" box, and the "4" in the subscript to the "total" box.

24 different ways

Combinations

Given a set of n **distinct objects**, any unordered subset of size k of the objects is called a **combination**.

Notation: $\binom{n}{k}$ or $C_{k,n}$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ **Read:** n chose k .

Note: $\binom{n}{n} = 1$, $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{k} = \binom{n}{n-k}$

$\binom{n}{k}$ = the number of different ways of selecting k objects from n distinct objects.

= the number of different ways of dividing n distinct objects into two subsets; one subset contains k objects and the other contains the rest $(n-k)$ objects. (**Order does not matter**)

Example: A boy has 4 beads – red, white, blue, and yellow. How different ways can **three** of the beads be chosen to trade away?



Solution: This is a combination since they are chosen without regard to order.

total

$$\binom{4}{3}$$

$$= \frac{4!}{3!(4-3)!} = \frac{4 \times 3!}{3!1!} = 4$$

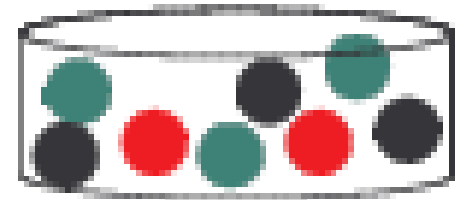
4 different ways

number selected

Example: Three balls are randomly selected at the same time (**without replacement**) from the jar below. Find the probability that **one** ball is **red** and **two** are **black**.

Solution:

$$N(S) = \binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8 \times 7 \times 6 \times 5!}{3!5!} = \frac{8 \times 7 \times 6}{3(2)} = 8 \times 7$$



Let $A = \{\text{one ball is red and two are black}\} \Rightarrow N(A) = \binom{2}{1} \binom{3}{2} = 2 \times 3$

$$\Rightarrow P(A) = \frac{N(A)}{N(S)} = \frac{2 \times 3}{8 \times 7} = \frac{3}{28}$$

without replacement

$P(A) = \{\text{one ball is red and two are black}\}$

$$\begin{aligned} P(R_1 B_2 B_3 \cup B_1 R_2 B_3 \cup B_1 B_2 R_3) &= \\ &= P(R_1 B_2 B_3) + P(B_1 R_2 B_3) + P(B_1 B_2 R_3) \end{aligned}$$

$$N(S) = 8 \times 7 \times 6$$

$$\begin{aligned} N(A) &= N(R_1 B_2 B_3 \cup B_1 R_2 B_3 \cup B_1 B_2 R_3) = \\ &= N(R_1 B_2 B_3) + N(B_1 R_2 B_3) + N(B_1 B_2 R_3) \\ &= 2 \times 3 \times 2 + 3 \times 2 \times 2 + 3 \times 2 \times 2 \\ &= 3(2 \times 3 \times 2) = 3 \times 2(3 \times 2) \end{aligned}$$



$$P(A) = \frac{N(A)}{N(S)} = \frac{3 \times 2(2 \times 3)}{8 \times 7 \times 6} = \frac{2 \times 3}{8 \times 7} = \frac{3}{28}$$

Example: A university warehouse has received a shipment of 25 printers, of which 10 are laser printers and 15 are inkjet models.

If 6 of these 25 are selected at random to be checked by a particular technician, *what is the probability that*

(1) *exactly 3 of these selected are laser printers?*

(2) *at least 3 inkjet printers are selected?*

Solution: Let $D_3 = \{\text{exactly 3 of the 6 selected are laser printers}\}$.

The experiment is selecting 6 printers from 25 (10 laser and 15 inkjet).

$$N = \binom{25}{6} = \frac{25!}{6!(25-6)!} = \frac{25!}{6!19!} \quad \leftarrow \quad \# \text{ selecting 6 printers from 25}$$

$$N(D_3) = \binom{15}{3} \binom{10}{3} = \frac{15!}{3!12!} \frac{10!}{3!7!} \quad \leftarrow \quad \begin{array}{l} \# \text{ selecting 3 printers from 15} \\ \text{and selecting 3 printers from 10} \end{array}$$

then

$$P(D_3) = \frac{N(D_3)}{N} = \frac{\frac{15!}{3!12!} \frac{10!}{3!7!}}{\frac{25!}{6!19!}} = 0.3083$$

Let $I_j = \{\text{exactly } j \text{ of the 6 selected are inkjet printers}\}, j = 3, 4, 5, 6$.

Then $I_3 = D_3$.

$$(2) P(\text{at least } 3 \text{ inkjet printers are selected}) = P(I_3 \cup I_4 \cup I_5 \cup I_6)$$

$$= P(I_3) + P(I_4) + P(I_5) + P(I_6)$$

$$= P(D_3) + P(I_4) + P(I_5) + P(I_6)$$

$$N(I_j) = \binom{15}{j} \binom{10}{6-j} = \frac{15!}{j!(15-j)!} \frac{10!}{(6-j)!(10-(6-j))!}, \quad j = 3, 4, 5, 6$$

 $P(\text{at least } 3 \text{ inkjet printers are selected}) =$

$$= \frac{\binom{15}{3} \binom{10}{3}}{\binom{25}{6}} + \frac{\binom{15}{4} \binom{10}{2}}{\binom{25}{6}} + \frac{\binom{15}{5} \binom{10}{1}}{\binom{25}{6}} + \frac{\binom{15}{6} \binom{10}{0}}{\binom{25}{6}} = .8530$$

2.4 Conditional Probability

Example :

Two machines produce the same type of products. Machine A produces 8, of which 2 are identified as defective. Machine B produces 10, of which 1 is defective. The sales manager randomly selected 1 out of these 18 for a demonstration.

- 1) What's the probability he selected product from machine A.
- 2) What's the probability that the selected product is defective?
- 3) What's the probability that the selected product is defective & from A?
- 4) If the selected product turned to be defective, what's the probability that this product is from machine A?

Solution :

1) $P(A) = 8/18=0.44$

2) $P(D) = 3/18=1/6$

3) $P(A \cap D) = 2/18$

4) $P(A | D) = 2/3=P(A \cap D)/P(D)$

		Condition		
		D	D'	
Machine	A	2	6	8
	B	1	9	10
		3	15	18

Conditional Probability

Definition:

For any two events A and B with $P(B) > 0$, the *conditional probability of A given that B has occurred* is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

From which we have:

$$P(A \cap B) = P(B) \cdot P(A|B)$$

or

$$P(A \cap B) = P(A) P(B|A)$$

Multiplication Rule for $P(A \cap B)$

Example

Suppose we have a fuse box containing 20 fuses of which 5 are defective (D) and 15 are non-defective (N). If 2 fuses are selected at random and removed from the box in succession without replacement, what is the probability that both fuses are defective?

Solution:

Let $D_j = \{\text{the } j\text{th fuse is } D\}, j = 1, 2.$

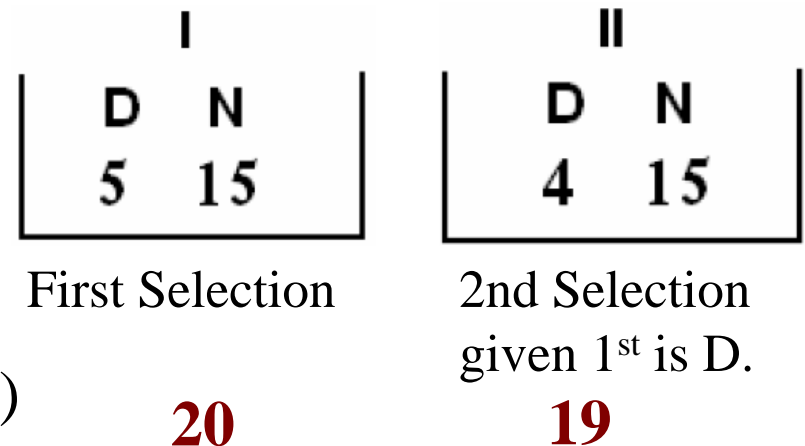
$\{\text{both fuses are defective}\} = D_1 \cap D_2$

$$\begin{aligned} P\{\text{both fuses are defective}\} &= P(D_1 \cap D_2) \\ &= P(D_1) P(D_2 | D_1) \end{aligned}$$

But $P(D_1) = 5/20$ and $P(D_2 | D_1) = 4/19$



$$P\{\text{both fuses are defective}\} = (5/20) (4/19) = 1/19 = 0.052632$$



Example

Four individuals will donate blood, if only the A+ type blood is desired and *only one* of these 4 people actually has this type, without knowing their blood type in advance, *if we select the donors randomly*, what's the probability that *at least three individuals must be typed* to obtain the desired type?

Solution:

Let $A_1 = \{\text{first type not A+}\}$ and $A_2 = \{\text{second type not A+}\}$. 

$$\begin{aligned} A_1 \cap A_2 &= \{\text{first type not A+ and second type not A+}\} \\ &= \{\text{at least three must be typed}\}. \end{aligned}$$

$$\begin{aligned} P\{\text{at least three must be typed}\} &= P(A_1 \cap A_2) \\ &= P(A_1) P(A_2 | A_1) \end{aligned}$$

But

$$P(A_1) = 3/4 \quad \text{and} \quad P(A_2 | A_1) = 2/3 \quad \img alt="green arrow" data-bbox="681 818 742 852"/>$$

$$P(A_1 \cap A_2) = (3/4) (2/3) = 2/4 = 1/2 = 0.5$$

The Law of Total Probability

If the events A_1, A_2, \dots, A_k be *mutually exclusive* and *exhaustive events*.
Then for any other event B ,

$$\begin{aligned} P(B) &= P(B | A_1) P(A_1) + \dots + P(B | A_k) P(A_k) \\ &= \sum_{i=1}^k P(B | A_i) P(A_i) \end{aligned}$$

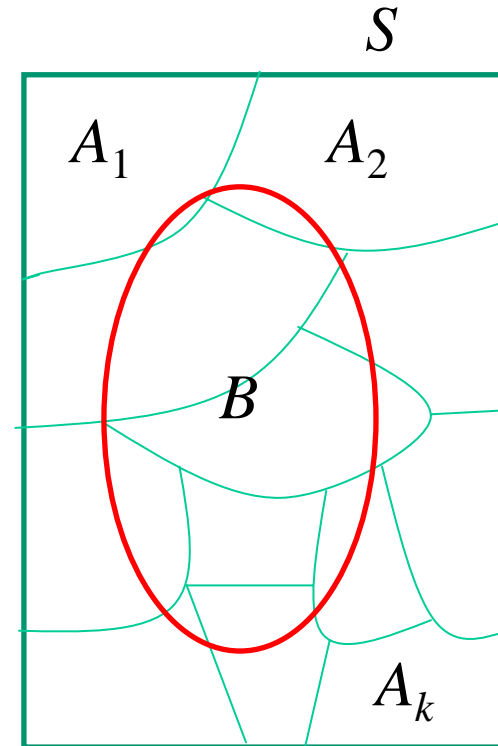
Proof:

$$B = (B \cap A_1) \cup \dots \cup (B \cap A_k)$$

$$P(B) = P((B \cap A_1) \cup \dots \cup (B \cap A_k))$$

$$= P(B \cap A_1) + \dots + P(B \cap A_k)$$

$$= P(B | A_1) P(A_1) + \dots + P(B | A_k) P(A_k)$$



Bayes' Theorem

Let A_1, A_2, \dots, A_n be a collection of k mutually exclusive and exhaustive events with $P(A_i) > 0$ for $i = 1, 2, \dots, k$. Then for any other event B for which $P(B) > 0$ given by

$$P(A_j | B) = \frac{P(B | A_j) P(A_j)}{\sum_{i=1}^k P(B | A_i) P(A_i)}, j = 1, 2, \dots, k \quad (1)$$

Proof:

We have

$$P(A_j | B) = \frac{P(B \cap A_j)}{P(B)}, j = 1, 2, \dots, k \quad (2)$$

but

$$P(B \cap A_j) = P(B | A_j) P(A_j) \quad (3)$$

and

$$P(B) = \sum_{i=1}^k P(B | A_i) P(A_i) \quad (4)$$

Substituting (3) and (4) into (2), we get (1).

Example:

A store stocks light bulbs from three suppliers. Suppliers A, B, and C supply 10%, 20% and 70% of the bulbs, respectively. It has been determined that company A's bulbs are 1% defective while company B's are 3% defective and company C's are 4% defective. *If a bulb is selected at random and found to be defective, what is the probability that it came from supplier B?*

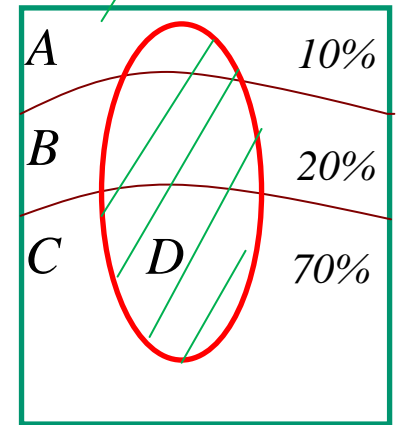
Solution: Let $D = \text{"defective"}$ We need $P(B|D)$?

We have

$$P(A) = 0.10, \quad P(D|A) = 0.01$$

$$P(B) = 0.20, \quad P(D|B) = 0.03$$

$$P(C) = 0.70, \quad P(D|C) = 0.04$$



$$\begin{aligned} P(B|D) &= \frac{P(D|B)P(B)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} \\ &= \frac{0.03 \times 0.2}{0.01 \times 0.1 + 0.03 \times 0.2 + 0.04 \times 0.7} = \frac{0.006}{0.035} = \frac{6}{35} = 0.171429 \end{aligned}$$

2.5 Independence

Independent Events

Two events A and B are independent if $P(A|B) = P(A)$.
Otherwise A and B are dependent.

Properties of independence

- ① Events A and B are independent $\iff P(A \cap B) = P(A) P(B)$
- ② $P(B|A) = P(B)$
- ③ If A and B are independent, then :
 - (1) A' and B
 - (2) A and B'
 - (3) A' and B' are all independent.

Question: A and B are mutually exclusive events, are they independent?

Example:

Consider a gas station with six pumps numbered 1,2,...,6 and let E_i denote simple event that a randomly selected customer uses pump i ($i = 1, 2, \dots, 6$). Suppose that $P(E_1) = P(E_6) = 0.10$, $P(E_2) = P(E_5) = 0.15$, and $P(E_3) = P(E_4) = 0.25$. Define events A, B, C by $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$ and $C = \{2, 3, 4, 5\}$. Compute $P(A)$, $P(A|B)$, $P(A|C)$?

Solution:

$$P(A) = P(E_2 \cup E_4 \cup E_6) = P(E_2) + P(E_4) + P(E_6) = 0.15 + 0.25 + 0.1 = 0.5$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(E_2)}{P(E_1) + P(E_2) + P(E_3)} = \frac{0.15}{0.50} = 0.3$$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(E_2) + P(E_4)}{P(E_2) + P(E_3) + P(E_4) + P(E_5)} = \frac{0.4}{0.8} = 0.5$$

$$P(A|B) \neq P(A)$$

A and B are dependent

$$P(A|C) = P(A)$$

A and C are independent

Compute $P(A')$, $P(A|B)$ and $P(A|C')$?

Independent of more than two events

Events A_1, \dots, A_n are mutually independent if for every k ($k=2,3,\dots,n$) and every subset of indices i_1, i_2, \dots, i_k ,

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \dots P(A_{i_k})$$

Note: In general

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2} | A_{i_1})P(A_{i_3} | A_{i_2} A_{i_1}) \dots \\ P(A_{i_k} | A_{i_{k-1}} \dots A_{i_2} A_{i_1})$$

Examples for Section 2.3 : Counting Techniques

- **Example1:** A house owner doing some remodelling requires the services of both a plumbing contractor and an electrical contractor; there are 12 plumbing contractors and 9 electrical contractors, in how many ways can the contractors be chosen?
- **Example 2:** A family requires the services of both an obstetrician and a pediatrician. There are two accessible clinics, each having two obstetricians and three pediatricians, family needs to select both doctor in the same clinic, in how many ways this can be done?
- **Example3:** There are 8 TA's are available, 4 questions need to be marked. How many ways for Prof. To choose 1 TA for each question? How many ways if there are 8 questions?

Example 4: In a box, there are 10 tennis balls labelled number 1 to 10

- 1) Randomly choose 4 with replacement
- 2) Choose 4 one by one without replacement
- 3) grab 4 balls in one time
- What is the probability the ball labelled as number 1 is chosen?

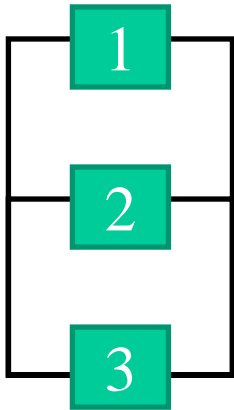
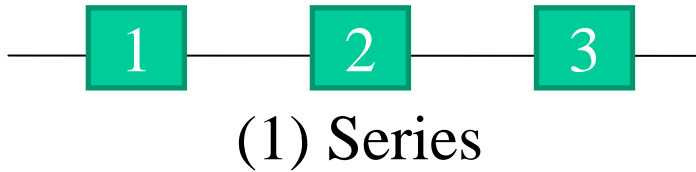
Example 5: A rental car service facility has 10 foreign cars and 15 domestic cars waiting to be serviced on a particular Sat. morning. Mechanics can only work on 6 of them. If 6 were chosen randomly, what's the probability that 3 are domestic 3 are foreign? What's the probability that at least 3 domestic cars are chosen?

Example 6: If a permutation of the word “white” is selected at random, find the probability that the permutation

- 1) begins with a consonant.
- 2) ends with a vowel.
- 3) has the consonant and vowels alternating.

Example : Consider three identical units. The probability that the lifetime of each exceeds t_0 is 0.9. Find the probability that the life time of the system consists of these three units exceeds t_0 if the system is:

(1) Series (2) Parallel.



Example

An executive on a business trip must rent a car in each of two different cities. Let A denote the event that the executive is offered a free upgrade in the first city and B represent the analogous event for the second city.

Suppose that $P(A) = .3$, $P(B) = .4$, and that A and B are independent events.

What is the probability that the executive is offered a free upgrade in at least one of the two cities?

If the executive is offered a free upgrade in at least one of the two cities, what is the probability that such an offer was made only in the first city?

If the executive is not offered a free upgrade in the first city, what is the probability of not getting a free upgrade in the second city? Explain your reasoning

Example:

339 physicians are classified as given in the table below. A physician is to be selected at random.

(1) Find the probability that:

- (a) the selected physician is aged 40 – 49
- (b) the selected physician smokes occasionally
- (c) the selected physician is aged 40 – 49 and smokes occasionally

(2) Find the probability that the selected physician is aged 40 – 49 given that the physician smokes occasionally.

		Smoking Habit			
		Daily (B_1)	Occasionally (B_2)	Not at all (B_3)	Total
Age	20 - 29 (A_1)	31	9	7	47
	30 - 39 (A_2)	110	30	49	189
	40 - 49 (A_3)	29	21	29	79
	50+ (A_4)	6	0	18	24
	Total	176	60	103	339

$N(S) = 339$ (equally likely outcomes)

Define the following events:

$A_3 =$ the selected physician is aged 40 – 49

$B_2 =$ the selected smokes occasionally.



$A_3 \cap B_2 =$ the selected physician is aged 40 – 49 and smokes occasionally

(1) (a) $A_3 =$ the selected physician is aged 40 – 49

$$P(A_3) = \frac{n(A_3)}{n(S)} = \frac{79}{339} = 0.2330$$

(b) $B_2 =$ the selected physician smokes occasionally

$$P(B_2) = \frac{n(B_2)}{n(S)} = \frac{60}{339} = 0.1770$$

(c) $A_3 \cap B_2 =$ the selected physician is aged 40 – 49 and smokes occasionally.

$$P(A_3 \cap B_2) = \frac{n(A_3 \cap B_2)}{n(S)} = \frac{21}{339} = 0.06195$$

(2) $A_3|B_2$ = the selected physician is aged 40 – 49 given that the physician smokes occasionally

$$(i) P(A_3 | B_2) = \frac{P(A_3 \cap B_2)}{P(B_2)} = \frac{0.06195}{0.1770} = 0.35$$

$$(ii) P(A_3 | B_2) = \frac{n(A_3 \cap B_2)}{n(B_2)} = \frac{21}{60} = 0.35$$

Notice that $P(A_3|B_2)=0.35 > P(A_3)=0.233$.

The conditional probability does not equal unconditional probability; i.e., $P(A_3|B_2) \neq P(A_3)$! What does this mean?

Questions: (1) $P(B_2 | A_3) = P(A_3 | B_2)$?

(2) Use the Law of Total Probability to compute $P(B_2)$.

(2) Use Bayes' theorem to compute $P(A_3 | B_2)$.